rules,[20] whose basis lies in the employment of broken $SU(3)$.

We conclude by considering the possible validity of our $\Delta T=2$ predictions. At present there appears to be little experimental evidence to weigh for or against our results. One analysis of the N^* mass splittings[21] indicates that the N^{*++}-N^{*0} mass difference is roughly -1 MeV, whereas a purely isovector splitting would make the N^{*++}-N^{*0} mass difference about -4 to -6 MeV in order to fit the observed N^{*-}-N^{*++} mass difference.[22] Therefore, our result for $\Delta T=2$ N^*-mass splittings appears to have the correct sign. The $NN\pi$, $N^*N\pi$, $N^*N^*\pi$ coupling constant shifts are also predominantly isotensor, and generally less than 5% of the $SU(2)$-invariant values. The most noteworthy feature of the coupling shifts is that all π^0 couplings to N, N^* are increased, those of $\pi^\pm$ decreased. This is consistent with a Fermi-Yang model of the pions, which obey the empirical inequality, $m(\pi^0)<m(\pi^\pm)$. In general, the isotensor predictions of the N, N^*, π model appear to have the correct qualitative properties, although we suspect on the basis of existing mass difference data that they are somewhat too large. We urge that a more complete phenomenological analysis of the N^* resonance be undertaken to provide us with a better understanding of the particle.

[20] S. Coleman and S. L. Glashow, Phys. Rev. **134**, B671 (1964).
[21] M. G. Olsson, Phys. Rev. Letters **14**, 118 (1965).
[22] A. H. Rosenfeld *et al.*, Rev. Mod. Phys. **39**, 1 (1967).

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Reggeization of Pion Exchange in Production Processes*

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Couplings of the pion Regge trajectory are discussed. We find that the kinematic factors prescribed by Wang must be supplemented by further kinematic terms. A simple physical interpretation is given for these additional terms. Our considerations lead to a model of pion trajectory couplings, which is in reasonable agreement with those experiments on vector- and tensor-meson production in which pion exchange is expected to dominate the forward peak.

I. INTRODUCTION

OVER the years a standard technique has evolved for the Reggeization of scattering amplitudes. First, one factors the helicity amplitude into the product of a function containing only dynamical singularities and zeros, and a known function $K(t)$ containing the kinematic singularities and zeros. Then one performs a Sommerfeld-Watson transformation on the dynamical function and expresses it in terms of Regge poles and other possible contributions in the J plane. The dynamical residue functions $\gamma(t)$ and trajectories $\alpha(t)$ are not known in general, but the step of factoring out the kinematical singularities has simplified their representation; as a first approximate attempt at parametrization, one commonly tries simple, smooth t dependences. With the existing prescription for kinematic singularities,[1] however, it has been found[2,3] that in production reactions where π dominance is suspected, very rapid variation of γ with t is required to fit the data at $t<0$ and the known pole strength at $t=m_\pi^2$.

In the present paper we study the Reggeization of the pion, including both the existing prescription[1] for kinematic singularities and further kinematic effects which were omitted previously. We find that the rapid variation of $\gamma(t)$ is caused by these further kinematic effects, rather than by dynamics. Our considerations suggest a model for $\gamma(t)$, and we illustrate its application to several processes.

We begin in Sec. II by describing the state of the art of extracting kinematic factors from helicity amplitudes. A quick review is given of the successive developments by Gell-Mann *et al.*,[4] Hara,[5] and Wang,[1] which were used in the analyses which indicated a rapid variation of $\gamma(t)$. The derivation of the further kinematic zeros, which we believe explain the rapid variation of $\gamma(t)$, is

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[1] L. L. Wang, Phys. Rev. **142**, 1187 (1966).
[2] R. Thews, Phys. Rev. **155**, 1624 (1967).
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[4] M. Gell-Mann, M. Goldberger, F. Low, E. Marx, and F. Zachariasen, Phys. Rev. **133**, B145 (1964).
[5] Y. Hara, Phys. Rev. **136**, B507 (1964).

discussed with special reference to the work of Cohen-Tannoudji *et al.*[6] and Fox.[7]

In Sec. III we introduce the physical interpretation of the kinematic factors which we will stress in treating the π trajectory, by showing explicitly how the results of Wang can be understood as simple threshold effects at the "pseudothresholds" as well as the normal thresholds. [For a t-channel reaction $a+b \to c+d$, the thresholds are at $t=(m_a+m_b)^2$ and $t=(m_c+m_d)^2$, and the pseudothresholds at $t=(m_a-m_b)^2$ and $t=(m_c-m_d)^2$. We show in Sec. III that it is helpful to think of the pseudothreshold as a threshold state of $a\bar{b}$ or $c\bar{d}$.] The further kinematic zeros uncovered by Fox,[7] Cohen-Tannoudji *et al.*,[6] and others have a similar physical interpretation. In a t-channel reaction such as $N\bar{N} \to N\bar{N}$, for example, the same J can couple to various l such as $l=J+1$ and $l=J-1$. At a threshold, the amplitude for $l=J+1$ has additional kinematic zeros relative to the amplitude for $l=J-1$.

The kinematic relations of Sec. II are exact properties of the helicity amplitudes. They do not, however, completely determine the residues $\gamma_\pi(t)$. The best we can do is use the relations, with the aid of the physical interpretation of Sec. III, as a guide in selecting approximate models for the residues. This is done in Sec. IV. Here we build up to our eventual choice of model, and illustrate how sensitive π trajectory exchange is to the proper choice of kinematic factors, by working out the numerical consequences of several models for $\pi N \to \rho\Delta$. This reaction is suitable for discussion because the density matrices at small t are consistent with π exchange dominance,[8] and the model of elementary π exchange with absorptive corrections[9] fits the data fairly well.[10] In all cases, the residue is factored into $K(t)\gamma(t)$, with $K(t)$ containing the kinematic singularities prescribed by Wang, and $\gamma(t)$ containing any additional kinematics as well as the dynamics.

The first approach given in Sec. IV is to approximate $\alpha_\pi(t)$ by a straight line with slope 1 GeV^{-2}, and set the reduced $\gamma(t)$ of the pion pole equal to a constant. Matching the constant to the known pion-pole strength at $t=m_\pi^2$, we find that the resulting cross section at $t<0$ is much too *low*. The cross section remains too low for flatter trajectories as long as we take γ as constant. We believe this approach gives too low a $d\sigma/dt$ because it fails to take into account the extra kinematic conditions particularly at $t=(M_\Delta-M_N)^2$.

The second approach is to consider elementary π exchange. Here, $\alpha_\pi(t)$ is fixed at zero, and $\gamma(t)$ turns out to be a rapidly varying function, vanishing at the thresholds and pseudothresholds in t and growing rapidly at negative t. This time the $\pi N \to \rho\Delta$ cross section at $t<0$ is much too *high*. The cross section still comes out too high if we return to the straight line $\alpha_\pi(t)$ with slope 1 GeV^{-2}, but retain the rapidly varying elementary $\gamma(t)$. The reason for the rapid variation of $\gamma_{\mathrm{elem}\,\pi}(t)$ is kinematic: the t-channel reaction $N\bar{\Delta} \to \pi\rho$ couples to an intermediate π state ($J^P=0^-$) only through $l>J$, whereas other states (such as $J^P=2^-$) can also couple through $l<J$; because of the higher orbital state, $\gamma_{\mathrm{elem}\,\pi}(t)$ acquires zeros at the thresholds and pseudothresholds. We believe that this approach gives too high a $d\sigma/dt$ because it incorporates the kinematic conditions in a somewhat inappropriate way: At thresholds far from $t=m_\pi^2$, the π trajectory is far from $\alpha=0$ and thus should escape the extra zeros which kinematics imposes on a spin-zero state.

Our final approach is to assume that along the trajectory, γ will behave approximately like $\gamma_{\mathrm{elem}\,\pi}$ at pseudothresholds or thresholds near $t=m_\pi^2$, where α_π is nearly zero, and will not behave anything like $\gamma_{\mathrm{elem}\,\pi}$ at pseudothresholds or thresholds far from $t=m_\pi^2$, where α_π is far from zero. This approach appears to handle the extra kinematic conditions in a reasonable way. By taking a stand on which thresholds are "far" from $t=m_\pi^2$ and which are "near," we obtain an approximate model of $\gamma(t)$ which fits the data.

Kinematic conditions affect the couplings of all trajectories. The existence and detection of the very rapid variation of $\gamma_\pi(t)$ in $\pi N \to \rho\Delta$, however, depends on the close conjunction of

(a) the pole with large residue at $t=m_\pi^2$,

(b) the pseudothreshold with kinematic zero at $t=(m_\Delta-m_N)^2=0.09$ GeV^2, and

(c) the physical region where the resulting residue variation can be observed at $t<0$.

This close conjunction of all three effects is a unique feature of π exchange.

In Sec. V we apply the same approach to $\pi N \to \rho N$, $KN \to K^*\Delta$, $\pi N \to f^0\Delta$, and $\pi N \to f^0 N$, in all of which π exchange is believed to dominate at small t. Good results (comparable to the peripheral model with absorptive corrections)[9] are obtained.

Besides the extra threshold relations, there are also relations among helicity amplitudes at $t=0$. In keeping with the spirit of our model, we have assumed that $\gamma_\pi(t)$ has the same behavior as $\gamma_{\mathrm{elem}\,\pi}(t)$ at $t=0$, and this is successful for the reactions named above.

Our model does not help in understanding some other reactions such as $np \to pn$ and $\gamma p \to \pi^+ n$. It is not clear whether this failure is due simply to lack of π-exchange

[6] G. Cohen-Tannoudji, A. Morel, and H. Navelet, Saclay Report, 1967 (unpublished). In addition to the crossing method described in our Sec. II, these authors present, and place their main emphasis upon, a powerful alternative method based on "transversity" amplitudes.

[7] G. C. Fox, Phys. Rev. **157**, 1493 (1967). More recently Fox has also worked out the general method based on "transversity" amplitudes (G. C. Fox, Ph.D. Thesis, Cambridge University, 1967, unpublished).

[8] D. Brown, G. Gidal, R. W. Birge, R. Bacastow, S. Y. Fung, W. Jackson, and R. Pu, University of California Radiation Laboratory Report No. 17665, 1967 (unpublished).

[9] For an over-all survey of this model, see J. D. Jackson, Rev. Mod. Phys. **37**, 484 (1965).

[10] Aachen-Berlin-CERN collaboration, Phys. Letters **19**, 608 (1965).

dominance in these reactions, or to conspiracy of the pion with other trajectories at $t=0$.

II. REVIEW OF METHODS FOR EXTRACTING KINEMATIC FACTORS

Our discussion of pion trajectory couplings will depend heavily on the identification of kinematic factors. It is therefore appropriate to begin with a review of how well such factors are understood.

We consider the reaction $a+b\rightarrow c+d$ in the channel where t is energy squared. Labeling helicity states by the same letters as the particles, the helicity amplitudes are $f_{cd;ab}{}^t$. The corresponding amplitudes for the reaction $b+\bar{d}\rightarrow c+\bar{a}$ in the channel where s is energy squared are $f_{c\bar{a};\bar{d}b}{}^s$. The cross section for the s channel reaction is

$$\frac{d\sigma}{d\Omega}=\frac{p_f}{(2\pi)^2 s p_i N_i}\sum_{\bar{d},b,\bar{c},a}|f_{c\bar{a};\bar{d}b}{}^s|^2, \tag{2.1}$$

where p_i and p_f are the initial and final center-of-mass (c.m.) momenta, and N_i is the number of initial spin states. In order to describe scattering in terms of exchanges, it is more convenient to rewrite $d\sigma/d\Omega$ in terms of t channel amplitudes. Using the fact that the crossing matrix for helicity amplitudes is orthogonal,[11] one finds

$$\frac{d\sigma}{d\Omega}=\frac{p_f}{(2\pi)^2 s p_i N_i}\sum_{ab,cd}|f_{cd;ab}{}^t|^2. \tag{2.2}$$

Now the f^t's contain various "kinematic" zeros and singularities, which are *fixed* by angular momentum requirements. Once these are identified and factored out, the remaining factors will have only dynamical singularities, with resulting simplifications in the representation of Regge residues.

The first step in this direction was taken by Gell-Mann *et al.*,[4] who isolated the kinematic zeros which occur at forward and backward scattering angles in the t channel. They wrote

$$f_{cd;ab}{}^t=(\sin\tfrac{1}{2}\theta_t)^{|\lambda-\mu|}(\cos\tfrac{1}{2}\theta_t)^{|\lambda+\mu|}\bar{f}_{cd;ab}{}^t, \tag{2.3}$$

where $\lambda=a-b$ and $\mu=c-d$ are the incoming and outgoing helicities. The kinematic factor $(\sin\frac{1}{2}\theta_t)^{|\lambda-\mu|}$ vanishes like $\theta_t{}^{|\lambda-\mu|}$ in the forward direction; this expresses the fact that conservation of angular momentum along the direction of forward scattering requires helicity conservation $\lambda=\mu$ or else the amplitude must vanish. Similarly, the kinematic factor $(\cos\frac{1}{2}\theta_t)^{|\lambda+\mu|}$ vanishes like $(\pi-\theta_t)^{|\lambda+\mu|}$ in the backward direction; this expresses the fact that conservation of angular momentum along the direction of backward scattering requires the helicity to reverse sign, $\lambda=-\mu$, or else the amplitude must vanish.

The remaining factor $\bar{f}^t$ has the partial-wave expansion

$$\bar{f}_{cd;ab}{}^t=\sum_J(2J+1)F_{cd;ab}{}^J(t)P_{J-M}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(cos\theta_t), \tag{2.4}$$

where

$$M=\text{maximum of }(|\lambda|,|\mu|),$$

and P_{J-M} is a Jacobi polynomial. Since the remaining $\cos\theta_t$ dependence in $\bar{f}^t$ is a sum over polynomials which are nonsingular at finite $\cos\theta_t$, and are not all zero at the same $\cos\theta_t$,[12] and since

$$\cos\theta_t=\frac{[2st+t^2-t\sum_i m_i^2+(m_a^2-m_b^2)(m_c^2-m_d^2)]}{\{[t-(m_a+m_b)^2][t-(m_a-m_b)^2][t-(m_c+m_d)^2][t-(m_c-m_d)^2]\}^{1/2}} \tag{2.5}$$

is linear in s, $\bar{f}^t$ has no further kinematic singularities or zeros in $\cos\theta_t$ or s.

The remaining problem is to factor out of $\bar{f}^t$ the kinematic singularities and zeros in the other variable, t. We know there are such singularities from any of the the following methods:

(i) study of Feynman diagrams;
(ii) study of behavior at t-channel thresholds;
(iii) relating f^t to f^s by crossing, and noting that f^s has kinematic singularities in $\cos\theta_s$ (i.e., in t) and that the crossing matrix has further singularities in t.

Hara[5] studied the singularities by a combination of all three methods, while Wang[1] used the crossing method exclusively. In either case, it proved most convenient to consider the "parity-conserving" combinations $\bar{f}_{cd;ab}{}^t\pm\bar{f}_{-c-d;ab}{}^t$, and Hara and Wang split these into a kinematic factor we shall call $K^\pm$ and a dynamical factor we shall call $\tilde{f}$:

$$\bar{f}_{cd;ab}{}^t\pm\bar{f}_{-c-d;ab}{}^t=K_{cd;ab}{}^\pm(t)\tilde{f}_{cd;ab}{}^{\pm t}(t,s), \tag{2.6}$$

giving a prescription for $K^\pm$.

In detail, the crossing method makes use of

$$f_i{}^t=\sum_j X_{ij}f_j{}^s, \tag{2.7}$$

where the subscript i stands for the set of helicity states cd, ab, and X_{ij} is the crossing matrix explicitly given by Trueman and Wick.[11] The corresponding relation for the

[11] T. L. Trueman and G. C. Wick, Ann. Phys. (N. Y.) **26**, 322 (1964).

[12] In particular, the Jacobi polynomials are finite and nonzero at $\cos\theta_t=\pm1$, which ensures that the half-angle factors in Eq. (2.3) contain just the right number of zeros at these points.

$\bar{f}$'s is

$$\bar{f}_i^t = \sum_j \bar{X}_{ij}\, \bar{f}_j^s\,, \tag{2.8}$$

where

$$\bar{X}_{ij} = \frac{X_{ij}(\sin\frac{1}{2}\theta_s)^{|\lambda'-\mu'|}(\cos\frac{1}{2}\theta_s)^{|\lambda'+\mu'|}}{(\sin\frac{1}{2}\theta_t)^{|\lambda-\mu|}(\cos\frac{1}{2}\theta_t)^{|\lambda+\mu|}}\,, \tag{2.9}$$

and λ' and μ' are the incoming and outgoing helicities in the s-channel c.m. system. The kinematic singularities and zeros of $\bar{f}^t$ in t are all in the explicitly known $\bar{X}$, since $\bar{f}^s$ has none. Thus by careful examination of $\bar{X}$, Wang was able to obtain a prescription for the kinematic singularities which is, as far as we know, complete. We will use Wang's prescription for the $K^\pm$ throughout this paper.

The part of the program which was *not* completed was to obtain a prescription for all kinematic zeros.[6,7,13] For example, if one takes Wang's singularity-free amplitudes $\tilde{f}_i^t$ for $NN \to NN$, it turns out they are not all linearly-independent at $t=0$ and $t=4M^2$. Thus, there are linear combinations of the $\tilde{f}_i^t$ which have kinematic zeros at these points. In terms of the equation

$$\tilde{f}_i^t = \sum_j \tilde{X}_{ij}\, \bar{f}_j^s\,, \tag{2.10}$$

this implies that the different rows of $\tilde{X}_{ij}$ are not all independent; i.e., the determinant of $\tilde{X}_{ij}$ has zeros, and the linear combinations of $\tilde{f}_i^t$ which vanish are the eigenvectors associated with the zero eigenvalues of $\tilde{X}$. The reason that Wang's study is incomplete, then, is that it is not enough to locate the singularities and zeros in *each row* of the matrix $\bar{X}_{ij}$; one must also locate the extra zeros of the *determinant* of $\tilde{X}_{ij}$.[14]

Of course, one can work out the determinant of $\tilde{X}$ by hand and locate its zeros explicitly in any particular case, but it would be much more convenient to have a general prescription for the answer. Fox[7] has given such a prescription for the class of reactions in which at least two particles are spinless. More recently, Cohen-Tannoudji, Morel, and Navelet[6] have pointed out that at zeros of $\det\tilde{X}$, elements of the inverse matrix $\tilde{X}^{-1}$ which appears in

$$\bar{f}_i^s = \sum_j \tilde{X}_{ij}^{-1}\, \tilde{f}_j^t \tag{2.11}$$

will have poles, and it is easier to locate poles than to form a determinant.[15] If the matrix elements of $\tilde{X}_{ij}^{-1}$ have the form

$$\tilde{X}_{ij}^{-1} = c_{ij}/(t-t_0) + (\text{terms regular at } t_0)\,, \tag{2.12}$$

then the condition

$$\sum_j c_{ij}\, \tilde{f}_j^t(t_0) = 0 \tag{2.13}$$

must hold, since neither $\bar{f}_i^s$ nor the individual elements $\tilde{f}_j^t$ have kinematic singularities or zeros at $t=t_0$. The derivation of kinematic conditions in the form (2.13) appears to be a very promising approach to the general problem of determining all kinematic zeros.[6]

The physical nature of the additional kinematic zeros may be guessed from the example of NN scattering. In this example, the conditions at $t=0$ are just the familiar relations which can be satisfied by vanishing of individual couplings or by conspiracy, and the conditions at $t=4M^2$ are those expected at threshold due to the dominance of ${}^3(l=J-1)_J$ over ${}^3(l=J+1)_J$ states at this point.[16]

In the meson-production reactions studied in the present paper, the additional kinematic zeros again have the physical interpretation of "conspiracy conditions" at $t=0$ and (as described in Sec. III) threshold and pseudothreshold conditions at $t=(m_a\pm m_b)^2$, $t=(m_c\pm m_d)^2$. It is the extra threshold and pseudothreshold zeros which provide the additional kinematic relations which we will use in discussing pion exchange. Concerning the kinematic conditions at $t=0$, we simply assume in the present paper that $\gamma(t)$ has the same behavior at $t=0$ as $\gamma_{\mathrm{elem}\,\pi}(t)$. This assumption is consistent with the experiments on meson production by pions.

III. THRESHOLD BEHAVIOR

Let us now return to the factors provided by Wang. They were obtained, we recall, by a rather abstract crossing argument [Eqs. (2.8) and (2.9)]. As Wang has noted,[1,17] they can be checked by less abstract arguments based on the threshold behavior of partial-wave helicity amplitudes with definite parity. Since the threshold argument increases one's confidence and will be used extensively in the following sections, we would like to work out an example showing how it goes at pseudothresholds as well as thresholds.

The example will be $\pi N \to \rho\Delta$, or rather the t-channel reaction $N\bar{\Delta} \to \rho\pi$. We take the amplitude $(f_{00;\frac{1}{2}\frac{1}{2}}{}^t + f_{00;-\frac{1}{2}-\frac{1}{2}}{}^t)$ which is purely $P=(-1)^{J+1}$ and is therefore contributed to by $N\bar{\Delta} \to \pi \to \rho\pi$. The Wang prescription for this amplitude is

$$f_{00;\frac{1}{2}\frac{1}{2}}{}^t + f_{00;-\frac{1}{2}-\frac{1}{2}}{}^t = K_{00;\frac{1}{2}\frac{1}{2}}{}^+(t)\,\tilde{f}_{00;\frac{1}{2}\frac{1}{2}}{}^t(s,t)\,, \tag{3.1}$$

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[13] B. Diu and M. Le Bellac, Orsay Report, 1967 (unpublished)

[14] Special considerations are also required for reactions involving photons; see, e.g., the discussion of photoproduction in S. Frautschi and L. Jones, Phys. Rev. **163**, 1820 (1967).

[15] No inversion of $\tilde{X}$ is required since $\tilde{X}^{-1}$ is known directly from the crossing relation between $\tilde{f}^t$ and $\bar{f}^s$.

[16] According to an investigation by Frank Henyey (private communication), if $\tilde{X}$ is defined as in Eq. (2.10), $\det\tilde{X}$ has three zeros at $t=4M^2$. These three zeros provide the amplitude for $[{}^3(J+1)_J \to {}^3(J+1)_J]$ with an additional factor $q_t^4 \sim (t-4M^2)^2$, and the amplitude for $[{}^3(J+1)_J \to {}^3(J-1)_J]$ with an additional factor q_t^2, relative to the amplitude for $[{}^3(J-1)_J \to {}^3(J-1)_J]$.

The determinant of the crossing matrix connecting the amplitudes used by Wang in Ref. 1, Sec. IV A, has two further zeros. The additional two zeros arise because Wang has not used "parity-conserving" helicity amplitude at this point; alternatively, they can be viewed as providing each of the ${}^1J_J$ and ${}^3J_J$ amplitudes with a threshold factor q_t^2 relative to the amplitude for $[{}^3(J-1)_J \to {}^3(J-1)_J]$.

[17] See also H. F. Jones, Imperial College London Report, 1966 (unpublished).

TABLE I. Threshold factors in the amplitude $(\bar{f}_{00;\frac{1}{2}\frac{1}{2}}{}^t+\bar{f}_{00;-\frac{1}{2}-\frac{1}{2}}{}^t)$ for $N\bar{\Delta}\to\rho\pi$.

Threshold	Lowest l	Possible J^P for lowest l	Allowed J^P with $P=(-1)^{J+1}$	Threshold factor
$t=(m_\Delta+m_N)^2$	$l_{\text{in}}=0$	1^-, 2^-	2^-	$[t-(m_\Delta+m_N)^2]^{-1}$
$t=(m_\rho+m_\pi)^2$	$l_{\text{out}}=0$	1^+	1^+	$[t-(m_\rho+m_\pi)^2]^{-1/2}$
$t=(m_\rho-m_\pi)^2$	$l_{\text{out}}=0$	1^+	1^+	$[t-(m_\rho-m_\pi)^2]^{-1/2}$
$=(m_\Delta-m_N)^2$	$l_{\text{in}}=0$	1^+, 2^+	1^+	$[t-(m_\Delta-m_N)^2]^{-1/2}$

where[1]

$$K_{00;\frac{1}{2}\frac{1}{2}}{}^+(t)=1/[t-(m_\Delta+m_N)^2]\{[t-(m_\Delta-m_N)^2]\times[t-(m_\rho+m_\pi)^2][t-(m_\rho-m_\pi)^2]\}^{1/2}. \quad (3.2)$$

Now, what factors in

$$\bar{f}_{cd;ab}{}^t=\sum_J(2J+1)F_{cd;ab}{}^J(t)\times P_{J-M}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta^t) \quad (3.3)$$

could explain these fixed singularities in t?

(i) At a *threshold*, $F^J(t)$ contains the usual factor $q_{\text{in}}{}^{l_{\text{in}}}q_{\text{out}}{}^{l_{\text{out}}}$, where

$$q_{\text{in}}=\{[t-(m_\Delta+m_N)^2][t-(m_\Delta-m_N)^2]\}^{1/2}/2(t)^{1/2}, \quad (3.4a)$$

and

$$q_{\text{out}}=\{[t-(m_\rho+m_\pi)^2][t-(m_\rho-m_\pi)^2]\}^{1/2}/2(t)^{1/2}. \quad (3.4b)$$

(ii) P_{J-M} is a polynomial in

$$\cos\theta_t=\frac{2st+t^2-t\sum m_i{}^2+(m_\Delta{}^2-m_N{}^2)(m_\rho{}^2-m_\pi{}^2)}{4tq_{\text{in}}q_{\text{out}}}. \quad (3.5)$$

The combination of these factors may give $\bar{f}^t$ an over-all singularity or zero at either of the two thresholds $t=(m_\Delta+m_N)^2$ and $t=(m_\rho+m_\pi)^2$.

To see how to obtain the over-all singularity at $t=(m_\Delta+m_N)^2$, consider $l_{\text{in}}=0$. For this orbital state, the $N\bar{\Delta}$ system has $J^P=1^-$ or 2^-. Since the particular amplitude we are considering has $P=(-1)^{J+1}$, the $N\bar{\Delta}$ system must have $J^P=2^-$. For $l_{\text{in}}=0$ there is no $q_{\text{in}}{}^{l_{\text{in}}}$ factor from F^J; for $J^P=2^-$ (and with $M=0$ for our particular example),

$$P_{J-M}{}^{(|\lambda-\mu|,|\lambda+\mu|)}(\cos\theta_t)=P_2{}^{(0,0)}(\cos\theta_t)$$

is a second-order polynomial in $\cos\theta^t$ with leading singularity $q_{\text{in}}{}^{-2}\sim[t-(m_\Delta+m_N)^2]^{-1}$ [Eqs. (3.5) and (3.4a)]. This agrees with Wang's result Eq. (3.2). For arbitrary l_{in}, F^J picks up an additional factor $q^{l_{\text{in}}}$ but the most singular part of P_{J-M} picks up a compensating factor $q^{-l_{\text{in}}}$, so the over-all singularity is unchanged. The key steps in this argument are tabulated in Table I, together with the corresponding steps which explain the factor at $t=(m_\rho+m_\pi)^2$.

Next we consider singularities at the *pseudothresholds*. It is helpful to think of the pseudothreshold $t=(m_\Delta-m_N)^2$ as a threshold involving the "antiparticle state" $E_N=-m_N$. Exactly the same arguments can be applied as above, except that the intrinsic parity of the antiparticle state is *reversed* relative to the particle state for a fermion (and *not reversed* for a boson). Thus, for $l_{\text{in}}=0$, the "N"$\bar{\Delta}$ system has $J^P=1^+$ or 2^+, and since the particular amplitude we are considering has $P=(-1)^{J+1}$, only $J^P=1^+$ contributes. For $l_{\text{in}}=0$ and $J=1$, F^J gives no $q_{\text{in}}{}^{l_{\text{in}}}$ factor and P_{J-M} gives $q_{\text{in}}{}^{-1}\sim[t-(m_\Delta-m_N)^2]^{-1/2}$ which agrees with Wang's result Eq. (3.2). Again, the argument is summarized in Table I together with the corresponding argument at $t=(m_\rho-m_\pi)^2$. Note that because of the parity change for fermions, the singularities are of different order at $t=(m_\Delta+m_N)^2$ and $t=(m_\Delta-m_N)^2$, whereas they are of the same order at $t=(m_\rho+m_\pi)^2$ and $t=(m_\rho-m_\pi)^2$.

The type of argument sketched above agrees with Wang's prescription in all cases we have checked.[18] We do not know of any corresponding physical argument for the behavior at $t=0$.

The additional kinematic zeros described by Fox,[7] Cohen-Tannoudji *et al.*,[6] and others can also be interpreted in terms of threshold effects[17] (except for the relations at $t=0$). We have already mentioned how this works out for $NN\to NN$ (see Ref. 16). In meson-production reactions, the corresponding relations are quite complicated, but simplifications occur in the special case of pure $J^P=0^-$ exchange which is relevant to our study of pion exchange.

For example, at $t=(M_\Delta-M_N)^2$ in $\pi N\to V\Delta$, the method of Cohen-Tannoudji *et al.*[6] yields one condition connecting the $\lambda=0$, $\mu=0$ amplitude to other helicity amplitudes, and other conditions not involving $\lambda=0$, $\mu=0$. In the special case of pure $J^P=0^-$ exchange, the first condition simplifies to $f_{\mu=0,\lambda=0}[t=(M_\Delta-M_N)^2]=0$. The physical reason for the simplification is that a spin-zero exchange couples exclusively to the $\lambda=0$, $\mu=0$ amplitude, so that the threshold conditions can only provide the $\lambda=0$, $\mu=0$ amplitude with extra zeros in this case rather than relating it to other amplitudes. Similar behavior occurs at the other pseudothresholds and thresholds.

Elementary pion exchange is a familiar case in which the extra zeros of pure 0^- exchange appear. It is represented by a Feynman diagram which automatically satisfies all the kinematic relations. The amplitude for elementary pion exchange in $\pi N\to\rho\Delta$ has the form

$$f_{00;\frac{1}{2}\frac{1}{2}}{}^t+f_{00;-\frac{1}{2}-\frac{1}{2}}{}^t=\frac{K^+(t)\gamma_{\text{el}}(t)}{t-m_\pi{}^2}, \quad (3.6)$$

[18] For amplitudes with $\lambda\neq0$, $\mu\neq0$, the combination $\bar{f}_{cd,ab}{}^t\pm\bar{f}_{-c-d,ab}{}^t$ no longer couples to definite parity states $[P=(-1)^J$ or $(-1)^{J+1}]$, so that the argument might appear to lose force. But for each l the *highest power* of $\cos\theta_t$ in P_{J-M} still has a definite parity, and it is the highest power which gives the controlling singularity.

TABLE II. Threshold factors in the elementary pion-exchange contribution to the amplitude $(f_{00;\frac{1}{2}\frac{1}{2}}{}^t+f_{00;-\frac{1}{2}-\frac{1}{2}}{}^t)$ for $N\bar{\Delta}\rightarrow\rho\pi$.

Threshold	J^P	Intrinsic spin and parity	l needed to make J^P	Factor for $K\gamma_\pi$	Factor for γ_π alone
$t=(m_\Delta+m_N)^2$	0^-	1^-, 2^-	$l_{\text{in}}=2$	$[t-(m_\Delta+m_N)^2]$	$[t-(m_\Delta+m_N)^2]^2$
$t=(m_\rho+m_\pi)^2$	0^-	1^+	$l_{\text{out}}=1$	$[t-(m_\rho+m_\pi)^2]^{1/2}$	$[t-(m_\rho+m_\pi)^2]$
$t=(m_\rho-m_\pi)^2$	0^-	1^+	$l_{\text{out}}=1$	$[t-(m_\rho-m_\pi)^2]^{1/2}$	$[t-(m_\rho-m_\pi)^2]$
$t=(m_\Delta-m_N)^2$	0^-	1^+, 2^+	$l_{\text{in}}=1$	$[t-(m_\Delta-m_N)^2]^{1/2}$	$[t-(m_\Delta-m_N)^2]$

where $K^+(t)$ has been given in Eq. (3.2). Evaluation of the Feynman diagram gives

$$\gamma_{\text{el}}(t)=c[t-(m_\Delta+m_N)^2]^2[t-(m_\Delta-m_N)^2]\times[t+(m_\rho+m_\pi)^2][t-(m_\rho-m_\pi)^2], \quad (3.7)$$

with the constant fixed by the known strength at the pole. The extra zeros provided by the workings of the kinematic relations for 0^- exchange are evident in (3.7).

To understand from an angular-momentum point of view why the kinematic factors in 0^- exchange, as exemplified by elementary pion exchange, differ from the Wang K^+, we observe that we could obtain the kinematic factor K^+ by setting $l=0$ and finding the corresponding J^P, whereas to obtain the elementary pion amplitude we set $J^P=0^-$ and find the corresponding l. For example, at $t=(m_\Delta+m_N)^2$ the intrinsic spin parity of $N\bar{\Delta}$ is 1^- or 2^-, so that to reach total $J^P=0^-$ requires an orbital state $l_{\text{in}}=2$. Then in the $J^P=0^-$ contribution to f^t (2.4), $F^{J=0}\sim q_{\text{in}}^2$ and $P_{J-M}=P_0=$ constant, so $f^t=(K^-\gamma_{\text{el}}/t-m_\pi^2)\sim q_{\text{in}}^2\sim[t-(m_\Delta+m_N)^2]$. This implies that γ_{el} picks up a factor $[t-(m_\Delta+m_N)^2]^2$, in agreement with Eq. (3.7). A resumé of the analogous arguments at the other thresholds is given in Table II.[19]

Repeating the same arguments for, say, an elementary 2^- particle, one sees that it would *not* have a rapidly varying $\gamma(t)$ in the $\lambda=0$, $\mu=0$ amplitude. The reason in the present language is that it can couple to $N\bar{\Delta}$ or $\pi\rho$ in both $l>J$ and $l<J$ states, and the contribution to $\gamma(t)$ from the *lowest* l does not vanish at threshold. Equivalently, one can say that a 2^- particle couples to several helicity amplitudes, and satisfies the threshold conditions by relations among these amplitudes rather than zeros in the $\lambda=0$, $\mu=0$ amplitude. The same is true for any particle on the sequence $(1^+, 2^-, 3^+, \cdots)$.[20] The rapid variation of $\gamma_{\text{el}}(t)$ for the pion, then, is a special feature of $J^P=0^-$, which can couple only to $\lambda=0$, $\mu=0$, and $l>J$, and therefore has extra threshold zeros in the $\lambda=0$, $\mu=0$ amplitude.

IV. MODELS FOR $\pi N\rightarrow\rho\Delta$

In this section we consider in succession three different models for pion exchange in $\pi N\rightarrow\rho\Delta$. The experimental density matrices for $\pi N\rightarrow\rho\Delta$ in the forward peak indicate that the non-helicity-change amplitude with $\mu=0=\lambda$ dominates.[8] It therefore makes sense to focus our attention on this amplitude, which is the one dissected in Sec. III.

Of the two possible $\mu=0=\lambda$ amplitudes $(f_{00;\frac{1}{2}\frac{1}{2}}{}^t\pm f_{00;-\frac{1}{2}-\frac{1}{2}}{}^t)$, the minus combination vanishes on account of parity conservation. The plus combination [Eq. (3.1)] is purely $P=(-1)^{J+1}$. Writing the Regge representation for $\tilde{f}_{00;\frac{1}{2}\frac{1}{2}}{}^t$, approximating it by the pion trajectory, and keeping only the leading power at large s, one obtains

$$(f_{00;\frac{1}{2}\frac{1}{2}}{}^t+f_{00;-\frac{1}{2}-\frac{1}{2}}{}^t)\xrightarrow[s\rightarrow\infty]{}K_{00;\frac{1}{2}\frac{1}{2}}{}^+(t)\gamma_\pi(t)\left(\frac{s}{s_0}\right)^{\alpha_\pi(t)}\times\frac{(1+e^{-i\pi\alpha_\pi(t)})}{\sin\pi\alpha_\pi(t)}\frac{\Gamma(\alpha_\pi+\frac{1}{2})(2\alpha_\pi+1)}{\Gamma(\alpha_\pi+1)}, \quad (4.1)$$

where K^+ was given in Eq. (3.2) and γ is the reduced residue function.

Our first model is a straight-line trajectory with constant dynamical coupling factor;

$$\alpha_\pi(t)=-0.02+t/\text{GeV}^2, \quad (4.2)$$

$$\gamma_\pi(t)=\text{const}. \quad (4.3)$$

We determine the constant from the known strength of pole at $t=m_\pi^2$, and take $s_0=2(m_\pi m_\rho m_N m_\Delta)^{1/2}=0.74$ GeV^2 (similar results would be obtained with the commonly assigned value $s_0=1$ GeV^2). The resulting forward peak [Fig. 1(b), curve (iii)] is much too sharp, falling far below the experimental cross section. The predicted cross section remains too low for models with flatter trajectories, as long as γ_π is held constant.

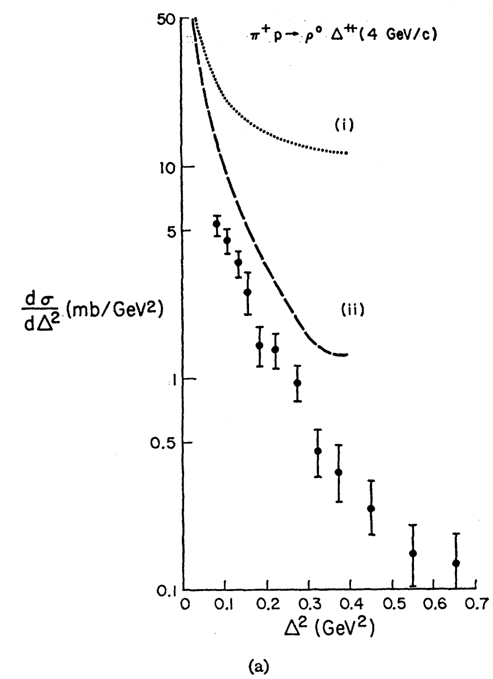
The second model we shall discuss is the elementary pion exchange as calculated, from the Feynman diagram. The amplitude for this model, and the coupling $\gamma_{\text{el}}(t)$, have already been given in Eqs. (3.6) and (3.7). As t

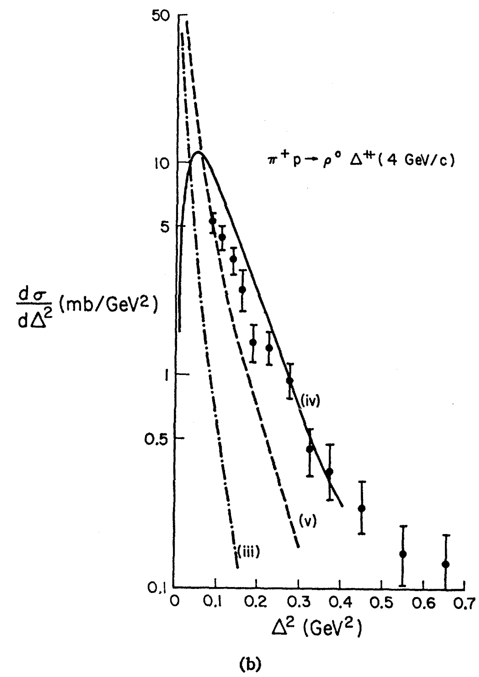
TABLE III. Location of thresholds and pseudothresholds of $N\bar{\Delta}\rightarrow\pi\rho$, and corresponding values of $\alpha_\pi(t)=-0.02+t/(1\text{ BeV}^2)$.

Threshold	t (BeV^2)	α_π
$t=(m_\Delta-m_N)^2$	0.09	0.07
$t=(m_\rho-m_\pi)^2$	0.37	0.35
$t=(m_\rho+m_\pi)^2$	0.79	0.77
$t=(m_\Delta+m_N)^2$	4.75	4.73

[19] As usual, we have no argument of this type for the behavior at $t=0$.

[20] Except for $J^P=1^+$ at $t=(m_\Delta+m_N)^2$, where $l=1$ is the lowest one can reach.





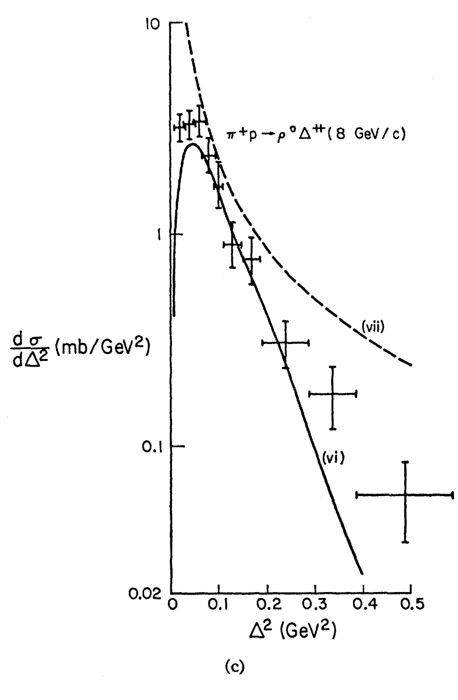


FIG. 1. (a) $\pi^+ p \to \rho^0 \Delta^{++}$ at 4 GeV/c. The data are from the Aachen-Berlin-Birmingham-Bonn-Hamburg-London (I.C.)-München collaboration (Ref. 21). The curves represent: (i) elementary OPE; and (ii) Reggeized OPE with kinematic factors of elementary exchange. (b) Comparison of the same data with: (iii) Reggized OPE with constant residue, kinematic factors from crossing matrix, (iv) Reggeized OPE with linear residue ($b = -0.005$ GeV^2), kinematic factors from crossing matrix, and (v) same model as (iv) with $b = 0.09$ GeV^2. (c) $\pi^+ p \to \rho^0 \Delta^{++}$ at 8 GeV/c. The data are from the Aachen-Berlin-CERN collaboration, Ref. 10. The curves represent: (vi) Reggeized OPE with linear residue ($b = -0.005$ GeV^2), kinematic factors from crossing matrix, and (vii) absorbed OPE (calculation in Ref. 10).

goes negative, $\gamma_{el}(t)$ is a rapidly rising function, which more than offsets $K(t)$ and produces much too *large* a cross section [Fig. 1(a), curve (i)]. The t dependence of γ_{el} is crucial here; even if we revert to the straight line α_π with slope 1 GeV^{-2}, the cross section remains too large as long as we retain $\gamma_{el}(t)$ [Fig. 1(a), curve (ii)].

One way of viewing the failure of elementary pion exchange is in terms of unitarity in the s channel. The elementary pion-exchange term does not include the influence of other channels, and even exceeds the unitarity bound in some low partial waves, so that absorptive corrections must be made. This approach has been used in the report of the Aachen-Berlin-CERN collaboration[10]; it produces theoretical results quite near the data [Fig. 1(c), curve (vii)]. The Reggeized pion exchange of our *first* model, however, lies well *below* the data, so that it is certainly below the unitarity bound. Some approach other than absorptive corrections is needed to improve the Regge model.

Another way of viewing the failure of our first two models is in terms of the t-channel threshold effects discussed at the end of Sec. III. The first model ignores these effects completely. The second model, on the other hand, incorporates the special effects of $J^P=0^-$ exchange even at thresholds where the Regge α is far from zero. These faulty approximations contrast with what is presumably the actual situation:

(i) The π trajectory couples mainly to the $\lambda=0$, $\mu=0$ amplitude, with the special threshold behavior of the elementary pion, at a pseudothreshold or threshold where $\alpha_\pi(t)$ is near enough to zero so that most of the contribution comes from the $J^P=0^-$ state.

(ii) When α_π is far from zero, the trajectory couples to various helicity amplitudes, and the threshold conditions are satisfied by relations among helicity amplitudes rather than by zeros in the $\lambda=0$, $\mu=0$ amplitude.

In our third model we attempt to incorporate kinematic effects into the $\lambda=0$, $\mu=0$ amplitude more realistically, by taking $\gamma(t)$ variable but using the extra zeros of elementary pion exchange only at pseudothresholds and thresholds where α_π is near zero. For simplicity we again confine our attention to the $\lambda=0$, $\mu=0$ amplitude. This, of course, restricts the validity of the model to small t (say, $|t|\lesssim 0.2$ GeV^2), but that is the region where π exchange is most prominent anyway.

We again take the straight-line approximation $\alpha_\pi(t)=-0.02+t/\text{GeV}^2$. The locations of the thresholds and pseudothresholds for $N\bar{\Delta}\to\rho\pi$, and the corresponding values of α_π, are listed in Table III. At $t=(m_\Delta-m_N)^2$, α_π is only 0.07 and the Regge amplitude projects heavily onto the 0^- partial wave. Thus the Regge amplitude has a 0^- part (probably dominant) which vanishes at $t=(m_\Delta-m_N)^2$, and another part (the projection onto other partial waves, probably rather small) which does not vanish. It is plausible to represent the net effect by a $(t-b)$ contribution to γ_{Regge}, where b is somewhat shifted from $(m_\Delta-m_N)^2$. At $t=(m_\rho+m_\pi)^2$ and $(m_\Delta+m_N)^2$, α_π is large, the projection onto higher partial waves whose coupling does not vanish at threshold is probably large, and there is no compelling reason to introduce another zero into γ_{Regge}. The point $t=(m_\rho-m_\pi)^2$ is intermediate and it is less clear what to do here.

These considerations lead us to the approximate form $\gamma_{\text{Regge}}=c(t-b)$ where b is small, and c is determined from the known strength at $t=m_\pi^2$. The cross sections computed in this way with $b=0.09$ GeV^2 and with $b=-0.005$ GeV^2 are compared with experiment[10,21] in Figs. 1(b) and 1(c). The fit with $b=(m_\Delta-m_N)^2=0.09$ GeV^2 [curve (v)] already represents a great improvement over the simple Regge model with constant γ [curve (iii)]. For smaller b, the fit becomes comparable to that obtained with absorptive corrections to elementary pion exchange [Fig. 1(c)]. By taking $b<m_\pi^2$, the "hook" in the experimental cross section is also obtained [curves (iv) and (vi)], although $b<m_\pi^2$ admittedly departs from the spirit in which we derived the model.

[21] Aachen-Berlin-Birmingham-Bonn-Hamburg-London (I.C.)-München Collaboration, Phys. Rev. 138, B897 (1965).

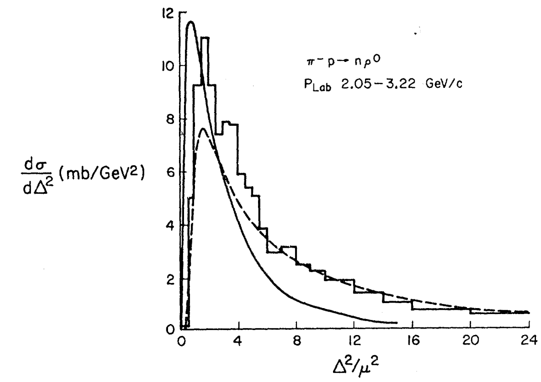


FIG. 2. $\pi^-p\to\rho^0n$ at 2.05–3.22 GeV/c. The data are from Jacobs (Ref. 24). The broken line is absorbed OPE (calculation from Ref. 24). The solid line is Reggeized OPE with kinematic factors from crossing matrix and factorization, constant residue.

V. COMPARISON OF THE MODEL WITH OTHER REACTIONS

The purpose of this section is to show that the model of pion residue functions developed for $\pi N\to\rho\Delta$ in Sec. IV also fits the differential cross sections for $\pi N\to\rho N$, $KN\to K^*\Delta$, $\pi N\to f^0\Delta$, and $\pi N\to f^0N$ at small t.

We begin with some comments which apply both to $\pi N\to\rho\Delta$ and to the reactions considered in the present section.

(i) A complete study of residue functions would involve a great deal more than comparison of differential cross sections. One would first separate the contributions of different helicity amplitudes to the cross section on the basis of density matrix elements and s and t dependence. Then individual residue functions for various exchanges could be isolated and studied. Some work along these lines has been done by Thews.[2] Our purpose in this paper, however, is only to indicate dominant features of the situation, and we shall not undertake such a detailed study.

(ii) For all reactions considered here, the density matrix elements and s dependence at small t are compatible with the assumption that pion exchange in the $\lambda=0$, $\mu=0$ amplitudes dominates. We show that the t dependence of $d\sigma/dt$ can also be understood on the basis of this assumption. The comparison with data must be confined to small t, however, because the assumption is expected to fail at $|t|\gtrsim 0.2$ GeV^2 on both

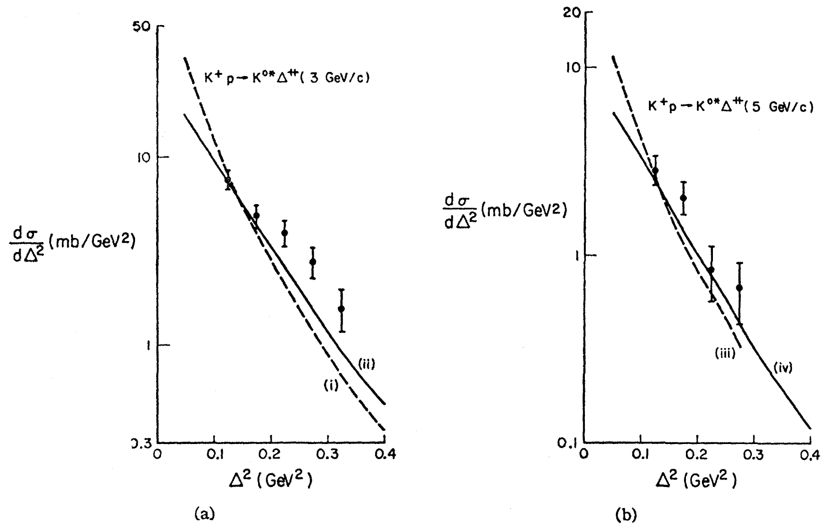


FIG. 3. (a) $K^+p \to K^{*0}\Delta^{++}$ at 3 GeV/c. The data are that of the CERN-Brussels collaboration, Ref. 25. The curves represent two possible fits with Reggeized OPE, kinematic factors from crossing matrix, and linear residue function. The solid line corresponds to $b=0.001$ GeV^2, the broken line to $b=0.045$ GeV^2. (b) The same reaction and models as 3(a), at 5 GeV/c.

theoretical and experimental grounds. Experiments[22] on the density matrices of produced ρ's, for example, show that for $|t|\gtrsim 0.2$ GeV^2 a large percentage of the vector mesons produced are transversely polarized, which implies $\mu\neq 0$. Thus our model for the $\lambda=0$, $\mu=0$ amplitude will be judged successful if it approximately fits $d\sigma/dt$ for $|t|\lesssim 0.2$ GeV^2 and tends to fall below the data for $|t|>0.2$ GeV^2, leaving room for other amplitudes.

(iii) To specify the model somewhat more fully than was done in Sec. IV, let us discuss the various terms in some detail. Under the assumption that $f_{\lambda=0,\mu=0}$ (π exchange) [Eq. (4.1)] dominates the forward peak, Eq. (2.2) takes the form

$$\frac{d\sigma}{dt}=1/4\pi[s-(m_1+m_2)^2][s-(m_1-m_2)^2] \times\left|K(t)\gamma_\pi(t)\left(\frac{s}{s_0}\right)^{\alpha_\pi(t)}\frac{(1+e^{-i\pi\alpha_\pi})}{\sin\pi\alpha_\pi} \times\frac{\Gamma(\alpha_\pi+\frac{1}{2})(2\alpha_\pi+1)}{\Gamma(\alpha_\pi+1)}\right|^2, \quad (5.1)$$

where m_1 and m_2 are the masses of the incident pseudoscalar and baryon for the reaction under consideration. The factor $\Gamma(\alpha_\pi+\frac{1}{2})/\Gamma(\alpha_\pi+1)$ is a coefficient of s^{α_π} in the asymptotic expansion of $P_\alpha(\cos\theta_t)$; the variation of these Γ functions and $(2\alpha_\pi+1)$ over the small range of t we are considering is not extremely important. The kinematic factor $K(t)$ for each reaction is taken from Wang[1,23] and listed in Table IV; for reactions of type $\pi+N\to(\rho$ or $f^0)+N$ it includes the factor t, which Wang derived[23] from the factorization condition. We use Eq. (4.2) (straight-line trajectory with slope 1 GeV^{-2}) for $\alpha_\pi(t)$. The reduced residue function $\gamma_\pi(t)$ suggested by our considerations in Sec. IV has the form $\gamma=$ const $(t-b)$ for the reactions involving an $N\bar{\Delta}$ vertex. A similar consideration of the reactions involving an $N\bar{N}$ vertex yields the form $\gamma=$ const, since in this case the nearby pseudothreshold has moved to $t=(m_N-m_{\bar{N}})^2=0$ and $\gamma_{\mathrm{el}\,\pi}(t)$ has no zeros at $t=0$. The value of each $\gamma_\pi(t)$ at $t=m_\pi^2$ is known from the experimental coupling strengths for $\rho\pi\pi$, $f^0\pi\pi$, $K^*K\pi$, $NN\pi$, and $\Delta N\pi$. We list these values in Table IV. The remaining parameter b has been treated as a variable, and we discuss below some of the values which give good fits. Finally, the scale factor s_0 has been chosen as $2(m_1m_2m_3m_4)^{1/2}$, which is not far from 1 GeV^2 for all cases considered. This choice is not crucial; fits with similar b's could be obtained for a wide range of s_0 (0.5 GeV$^2\lesssim s_0\lesssim 5$ GeV^2).

Having described the model, we turn now to the individual reactions.

$\pi^+p\to\rho^0\Delta^{++}$. The cross section for $b=(m_\Delta-m_N)^2=0.09$ GeV^2 is shown in Fig. 1(b) [curve (v)]. The best fit, which gives the hook visible in the 8(GeV/c data in

TABLE IV. The Wang kinematic factor $K(t)$, and the strength of the dynamical residue function $\gamma_\pi(t)$ at $t=m_\pi^2$, for pion trajectory exchange in various reactions. The symbol τ_{ij} denotes $\{[t-(m_i+m_j)^2][t-(m_i-m_j)^2]\}^{1/2}$.

Reaction	$K(t)$	$\gamma_\pi(t=m_\pi^2)/$ (mb)$^{1/2}$
$\pi^+p\to\rho^0\Delta^{++}$	$\tau_{\rho\pi}^{-1}\tau_{N\Delta}^{-1}[t-(m_N+m_\Delta)^2]^{-1/2}$	4.5 GeV^6
$\pi^-p\to\rho^0n$	$\sqrt{t}(\tau_{\rho\pi}^{-1})$	9.9 GeV^2
$K^+p\to K_0{}^*\Delta^{++}$	$\tau_{KK^*}^{-1}\tau_{N\Delta}^{-1}[t-(m_N+m_\Delta)^2]^{-1/2}$	2.5 GeV^6
$\pi N\to f^0N$	$\sqrt{t}(\tau_{\pi f}^{-2})$	67.2 GeV^4
$\pi^+p\to f^0\Delta^{++}$	$\tau_{\pi f}^{-2}\tau_{N\Delta}^{-1}[t-(m_N+m_\Delta)^2]^{-1/2}$	30.5 GeV^8

[22] D. H. Miller *et al.*, Phys. Rev. **153**, 1423 (1967); W. L. Yen *et al.*, Phys. Rev. Letters **18**, 1091 (1967).
[23] L. L. Wang. Phys. Rev. **153**, 1664 (1967).

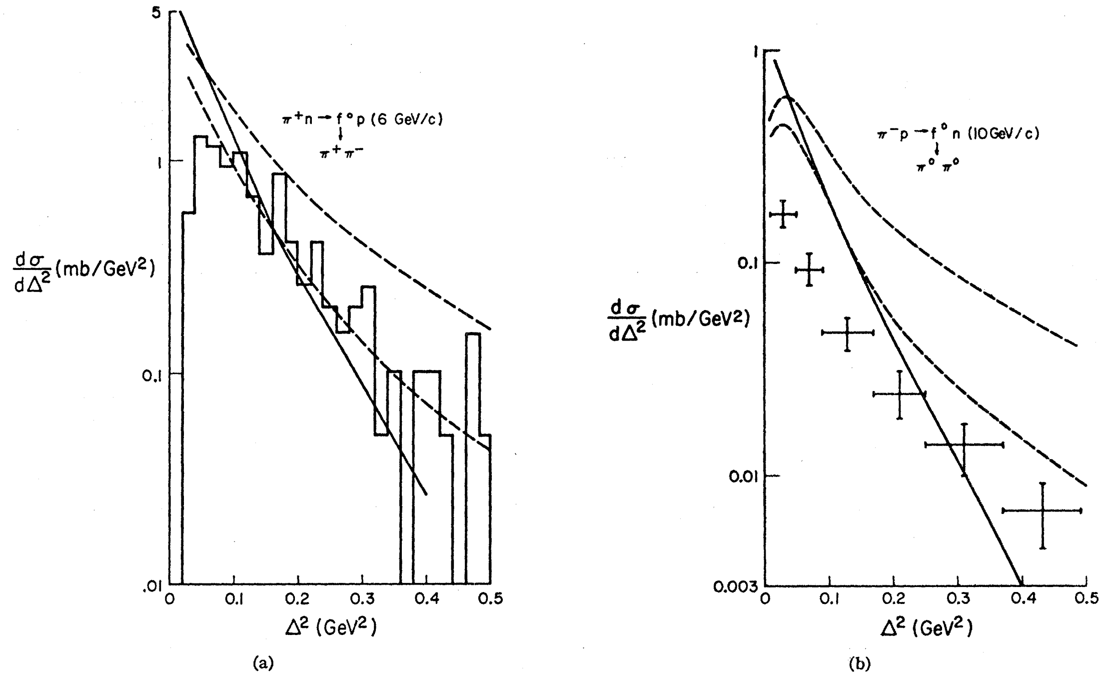


FIG. 4. (a) $\pi^+n \to f^0p$ at 6 GeV/c. The data are from Bruyant *et al.*, Ref. 26. The broken lines are various absorption model predictions (calculations by Yock and Gordon, Ref. 3). The solid line represents Reggeized OPE with kinematic factors from crossing matrix and factorization, constant residue. (b) $\pi^-p \to f^0n$ at 10 GeV/c. The data are from Wahlig *et al.*, Ref. 27. The curves have the same significance as in 4(a).

Fig. 1(c) as well as at lower energies,[8] is given by $b=-0.005$ GeV² [curves (iv) and (vi)].

$\pi^-p \to \rho^0 n$. The cross section of form (5.1), with K and γ_π from Table IV, fits the data[24] quite well (Fig. 2) without additional parameters.

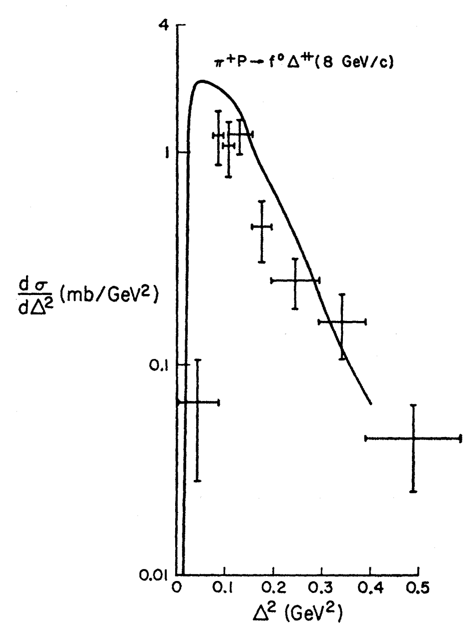


FIG. 5. $\pi^+p \to f^0\Delta^{++}$ at 8 GeV/c. The data are from the Aachen-Berlin-CERN collaboration, Ref. 10. The curve represents Reggeized OPE with kinematic factors from crossing matrix, linear residue ($b=-0.015$ GeV²).

$K^+p \to K_0^*\Delta^{++}$. The differential cross section for this reaction in the extreme forward direction is not well known at present. As a result, there are two solutions of the form indicated above which fit the data points[25]—one solution with $b=0.045$ GeV² and the other with $b=0.001$ GeV². They are both plotted in Fig. 3.

$\pi N \to f^0 N$. As in the analogous reaction $\pi N \to \rho^0 N$, we find that the cross section[26,27] can be fit quite well by use of a constant residue function with the kinematic factor and $\gamma_\pi(t=m_\pi^2)$ of Table IV (Fig. 4).

$\pi^+p \to f^0\Delta^{++}$. Fitting this reaction with a linear residue function we find, as in the case of $\pi N \to \rho^0\Delta$, that the zero must fall at $t<0$ ($b=-0.015$) in order to reproduce the hook at low t (Fig. 5). Again, a fit to the magnitude outside the hook region can be obtained with a zero at the $N\bar{\Delta}$ pseudothreshold.

VI. DISCUSSION

Mandelstam[28] has noticed a difficulty which arises in the formal Reggeization of processes involving particles

[24] L. D. Jacobs, Ph.D. thesis, University of California Radiation Laboratory Report No. 16877, 1966, p. 145 (unpublished).

[25] CERN-Brussels collaboration, Nuovo Cimento **46A**, 593 (1966); and Y. Goldschmidt-Clermont (private communication).

[26] F. Bruyant *et al.*, in *Proceedings of the International Conference on High-Energy Physics, Dubna, 1964* (Atomizdat, Moscow, 1965), Vol. 1, p. 422.

[27] M. Wahlig *et al.*, Phys. Rev. **147**, 941 (1966).

with higher spins. Normal methods of continuation in angular momentum produce, for example, contributions to the Regge residues for $\pi N \to VN$ from $l=-1$ states. This is in contrast to our arguments, in which only physical orbital angular momenta $l \geq 0$ in the t channel have been considered. Mandelstam has also demonstrated,[28] however, that the contributions from unphysical l may be cancelled by other terms, if all of the external particles involved lie on Regge trajectories. Thus our restriction to physical orbital angular momenta conforms to the result one presumably obtains by consistently Reggeizing all hadrons both internal and external.

Dashen and Frautschi[29] have predicted from static model bootstrap calculations that

$$\beta_{\pi N\bar{N}}(t)/\beta_{\pi N\bar{\Delta}}(t) \approx \text{const.} \tag{6.1}$$

This relation has been tested in various contexts[30,31] and found to be in agreement with vector-meson production data. It is interesting to note that Eq. (6.1) would not be consistent with β's deduced purely from kinematic factors, but is roughly consistent at negative t with the behavior we have found from angular-momentum considerations,

$$\beta_{\pi N\bar{N}} \sim \sqrt{t} \quad \text{and} \quad \beta_{\pi N\bar{\Delta}} \sim (t-b)/[t-(m_\Delta - m_N)^2]^{1/2}.$$

The approach we have discussed does not solve all problems connected with pion exchange. For example, the energy dependence of $np \to pn$ is consistent with a highest α close to α_π, so it is possible that pion exchange is important here. But our methods with the no-conspiracy assumption give a result similar to elementary pion exchange, with a dip in the forward direction rather than the observed "spike." As has often been conjectured, this may be a case where conspiracy is important.

Another reaction in which pion exchange is generally believed to be important is $\gamma p \to \pi^+ n$. Here again, the no-conspiracy assumption produces a forward dip in the one-pion-exchange (OPE) amplitude, whereas it has long been known in low-energy photoproduction that the data have no such dip, and recent high-energy experiments[32] down to $\theta_{\text{c.m.}} = 2.5°$ similarly show no sign of a dip. This strongly suggests that a conspiracy has altered the $\sqrt{t}$ behavior of some residue functions, possibly including that of the pion. We have discussed the details of this case in another paper.[33]

Note added in proof. For the reactions in which a Δ is produced, we have seen that phenomenological fits to our model yield $\gamma_\pi \sim t$ or $\gamma_\pi \sim (t-0.09 \text{ GeV}^2)$ depending on whether or not $d\sigma/dt$ hooks over and has a sharp dip at the smallest $|t|$ observed. As noted at the end of Sec. IV, our pseudothreshold interpretation is compelling only in the latter case [no hook; $\gamma_\pi \sim (t-0.09 \text{ GeV}^2)$]. It has been pointed out to us by M. Le Bellac and J. D. Jackson that the hook observed in some data [e.g., Fig. 1(c) and 5] may be an artificial effect caused by finite resonance widths. For example, to measure Δ production the experimentalist normally integrates πN production over the whole Δ peak. For forward production, however, $t_{\min}$ (corresponding to $\theta_s = 0^0$) depends on the πN (mass)^2, and the counts recorded at the lowest t are produced by only the low-energy tail of the Δ resonance. Thus it is natural for the counts to hook over and dip at the smallest t, even if the intrinsic cross section for producing Δ's does not. If this is indeed the reason for the hook and forward dip, then $\gamma_\pi \sim (t-0.09 \text{ GeV}^2)$ is the correct fit and our pseudothreshold interpretation is confirmed. On the other hand, if the hook persists even after the data are corrected for finite width effects, then the proposal by M. Le Bellac [Argonne Report, 1967 (unpublished)] relating $\gamma_\pi \sim t$ to conspiracy effects becomes a possible alternative (applications of this model should still take pseudothreshold conditions into account, however). We hope that reanalysis of the existing data will provide an answer to this question in the near future.

Δ production by Reggeized pion exchange has also been investigated recently by B. Haber, U. Maor, G. Yekutieli, and E. Gotsman [Weizmann Institute Report, 1967 (unpublished)]. They use $\gamma_{\text{el}}(t)$ [our Eq. (3.7)] and counteract the rapid growth of $\gamma_{\text{el}}(t)$ in the low $|t|$ region by giving α_π an unusually large slope [$\alpha_\pi'(t) \sim 1.75/\text{GeV}^2$]. They find that this model gives fits which are good at small t, inadequate at large t.

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